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Tricks for Multivariable Limits

I) Sertöz Theorem : (Apply on scratch paper!)

If the limit of a multivariable function is given in the following form :

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^c + y^d}$$

i) If c and d are positive even integers

ii) If $\frac{a}{b} + \frac{c}{d} > 1$

Then the limit will exist and is equal to 0.If the second condition is not satisfied then for sure the limit will not exist.

** Note : This method is not allowed in exams, but you can use it on a scratch paper to know if the limit will exist or not. **

II) Method A :

If the limit of a multivariable function is given in the following form :

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^c + y^d}$$

i) If $a = c$ and/or $b = d$

ii) If **Sertöz Theorem** fails at least one of its conditions

Then Limit does NOT exist

Use of **Method A** :

Consider the following multivariable limit :

$$\lim_{(x,y)\to(0,0)} \frac{xy}{8x+4y} \qquad (2018\ Spring\ Exam\ 2\ )$$

i) This function has the form of :

$$\frac{x^a y^b}{x^c + y^d} \quad (ignore\ the\ coeffeciants)$$

Where $a = c$ and $b = d$

ii) This function fails **Sertöz Theorem** because even though

$\frac{a}{b} + \frac{c}{d} = \frac{1}{1} + \frac{1}{1} = 2 > 1$, c and d are NOT EVEN integers

Now we will present the path used to disprove the existence of this limit.

Again, **on scratch paper** :

Set the function equal to m :

$$\frac{xy}{8x+4y} = m$$

Find y as a function of x :

$$xy = m(8x+4y)$$

$$xy = 8mx + 4my$$

$$4my - xy = -8mx$$

$$y(4m - x) = -8mx$$

$$y = \frac{-8mx}{4m - x}$$

$$\boldsymbol{y = \frac{8mx}{x - 4m}}$$

Now on your **exam paper** you tell him consider the path :

$$y = \frac{8mx}{x - 4m}$$

We will substitute this path in the multivariable limit, and we will get :

$$\lim_{x \to 0} \frac{x\left(\frac{8mx}{x - 4m}\right)}{8x + 4\left(\frac{8mx}{x - 4m}\right)}$$

$$\lim_{x \to 0} \frac{\frac{8mx^2}{x - 4m}}{8x + \frac{32mx}{x - 4m}}$$

$$\lim_{x \to 0} \frac{\frac{8mx^2}{x - 4m}}{\frac{8x(x - 4m) + 32mx}{x - 4m}}$$

$$\lim_{x \to 0} \frac{\frac{8mx^2}{x - 4m}}{\frac{8x^2 - 32mx + 32mx}{x - 4m}}$$

$$\lim_{x \to 0} \frac{\frac{8mx^2}{x - 4m}}{\frac{8x^2}{x - 4m}} = \boldsymbol{m}$$

The limit depends on $\boldsymbol{m}$ **and only** $\boldsymbol{m}$ which means that the limit does NOT exist

Example II :

Consider the following multivariable limit :

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{7x^3+y^4}$$

By checking the conditions, we will see that one of the conditions of **Sertöz Theorem** is not satisfied and this limit has $a = c$ so we are sure that the limit does not exist, and we must prove it :

On scratch paper set function equal to m

$$\frac{x^3y^2}{7x^3+y^4} = m$$

$$x^3y^2 = m(7x^3+y^4)$$

$$x^3y^2 = 7mx^3 + my^4$$

$$x^3y^2 - 7mx^3 = my^4$$

$$x^3(y^2-7m) = my^4$$

$$x^3 = \frac{my^4}{y^2-7m}$$

$$\boldsymbol{x = \sqrt[3]{\frac{my^4}{y^2-7m}}}$$

On exam paper you state that we will consider the following path :

$$x = \sqrt[3]{\frac{my^4}{y^2-7m}}$$

Now we will substitute in the limit :

$$\lim_{y\to 0}\frac{\left(\sqrt[3]{\frac{my^4}{y^2-7m}}\right)^3 y^2}{7\left(\sqrt[3]{\frac{my^4}{y^2-7m}}\right)^3+y^4}$$

$$\lim_{y\to 0}\frac{\frac{my^4}{y^2-7m}y^2}{7\frac{my^4}{y^2-7m}+y^4}$$

$$\lim_{y\to 0}\frac{\frac{my^6}{y^2-7m}}{\frac{7my^4+y^4(y^2-7m)}{y^2-7m}}$$

$$\lim_{y\to 0}\frac{\frac{my^6}{y^2-7m}}{\frac{7my^4-7my^4+y^6}{y^2-7m}}$$

$$\lim_{y\to 0}\frac{\frac{my^6}{y^2-7m}}{\frac{y^6}{y^2-7m}}=\boldsymbol{m}$$

The limit depends on $\boldsymbol{m}$ **and only** $\boldsymbol{m}$ which means that the limit does NOT exist

III) Method B :

If the limit of a multivariable function is given in the following form :

$$\lim_{(x,y)\to(0,0)} \frac{x^a + y^b}{x^c + y^d}$$

i) If we only know that $a > c$ and $b > d$

Then Limit Exists and is equal to 0

Use of **Method B** :

Consider the following multivariable limit :

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^4}{x^2 + y}$$

What I will do is attempt to make the numerator equal to its denominator so we will factor the numerator :

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^4}{x^2 + y} = \lim_{(x,y)\to(0,0)} \frac{x^2(x) + y(y^3)}{x^2 + y}$$

We will now use a very important inequality we will state that :

$$\mathbf{0} \leq \lim_{(x,y)\to(0,0)} \left|\frac{x^2(x) + y(y^3)}{x^2 + y}\right| \leq \lim_{(x,y)\to(0,0)} \left|\frac{(x^2 + y)(x + y^3)}{x^2 + y}\right|$$

Hence , we will obtain :

$$\mathbf{0} \leq \lim_{(x,y)\to(0,0)} \left|\frac{x^3 + y^4}{x^2 + y}\right| = \lim_{(x,y)\to(0,0)} \left|\frac{x^2(x) + y(y^3)}{x^2 + y}\right| \leq \lim_{(x,y)\to(0,0)} |x + y^3| = \mathbf{0}$$

Why does this inequality work?

I will prove that :

$$x^a x^b + y^c y^d \ \leq (x^a + y^c)(x^b + y^d)$$

Simply expand :

$$x^a x^b + y^c y^d \ \leq x^a x^b + x^a y^d + y^c x^b + y^c y^d$$

$$\boldsymbol{x^a x^b + y^c y^d \ \leq x^a x^b + y^c y^d + x^a y^d + y^c x^b}$$

$$\text{But } x^a y^d + y^c x^b \geq 0$$

$$Hence : x^a x^b + y^c y^d \ \leq (x^a + y^c)(x^b + y^d)$$

IV) Method C :

If the limit of a multivariable function is given in the following form :

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^a}{x^{a+1} + y^{a+1}}$$

i) If a is a positive **even** integer or if $a + 1$ is a positive **odd** number

Then Limit does NOT exist

We will take 2 different paths to disprove the existence of such limit :

a) First Path : $\boldsymbol{x = 0}$

b) Second Path : $\boldsymbol{y = (x^{2a} - x^{a+1})^{\frac{1}{a+1}}}$

Use of **Method C** :

Consider the following multivariable limit :

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$$

Consider the First Path : $x = 0$ then

$$\lim_{y\to 0} \frac{0^2(y^2)}{0^3+y^3} = 0$$

Consider the Second Path : $y = (x^4 - x^3)^{\frac{1}{3}}$ then

$$\lim_{x\to 0} \frac{x^2(x^4-x^3)^{\frac{2}{3}}}{x^3+(x^4-x^3)^{\frac{3}{3}}}$$

$$\lim_{x\to 0} \frac{x^2[(x-1)(x^3)]^{\frac{2}{3}}}{x^4}$$

$$\lim_{x\to 0} \frac{x^2x^2(x-1)^{\frac{2}{3}}}{x^4} = 1$$

Since the limit is equal to 0 from the first path and the limit equals to 1 from the second path , we can conclude that the limit does not exist

Example II :

Consider the following multivariable limit :

$$\lim_{(x,y)\to(0,0)} \frac{x^6y^6}{x^7+y^7}$$

First Path would be $x = 0$ which would yield the limit to 0

Second Path would be $y = (x^{12} - x^7)^{\frac{1}{7}}$ which would given

$$\lim_{x\to 0} \frac{x^6(x^{12} - x^7)^{\frac{6}{7}}}{x^{12} + x^7 - x^7}$$

$$\lim_{x\to 0} \frac{x^6[(x^5 - 1)(x^7)]^{\frac{6}{7}}}{x^{12} + x^7 - x^7}$$

$$\lim_{x\to 0} \frac{x^6x^6(x^5 - 1)^{\frac{6}{7}}}{x^{12}}$$

$$\lim_{x\to 0} (x^5 - 1)^{\frac{6}{7}} = 1$$

Since the limit is equal to 0 from the first path and the limit equals to 1 from the second path , we can conclude that the limit does not exist